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ON CERTAIN CONSERVATION PROPERTIES IN GAS DYNAMICS*

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A previously unknown invariant of the vertex lines of a stationary barotropic, ideal gas flow is discovered, An analogue of this invariant and of other invariants of the stream and vortex lines is obtained for the more general case of non-barotropic flow.

An equation is obtained describing the variation in the projection of the vorticity on the direction of the velocity in three-dimensional ideal gas flow. Examples are shown where the projection does not vary along the stream lines, and this yields an additional integral of the gas-dynamic equations.

1. Consider the steady flow of an ideal compressible gas. We denote the velocity vector by $\mathbf{v}$, $\boldsymbol{\omega} = \operatorname{rot} \mathbf{v}$ is the vorticity, p is the pressure and ρ is the density. In gas-dynamics the quantities conserved along the stream lines (stream line invariants) are of interest. We known, in particular, that along with the entropy σ the Ertel vortex potential $E_0 = (\boldsymbol{\omega}\cdot\nabla\sigma)/\rho$ is also conserved along the stream lines.

In a barotropic gas flow /1–3/ $E_\lambda = (\boldsymbol{\omega}\cdot\nabla\lambda)/\rho$ serves as the stream line invariant

$$\mathbf{v}\cdot\nabla\left(\frac{\boldsymbol{\omega}\cdot\nabla\lambda}{\rho}\right) = 0 \tag{1.1}$$

where λ is an arbitrary function, constant along the stream lines

$$\mathbf{v}\cdot\nabla\lambda = 0 \tag{1.2}$$

Relations (1.1), (1.2) express the Euler-Ertel theorem /1/ for a compressible barotropic gas.

We must also establish the invariants of the vortex lines. The Bernoulli function H represents one of these invariants:

$$\boldsymbol{\omega}\cdot\nabla H = 0, \quad H = \frac{q^2}{2} + \int\frac{dp}{\rho(p)}, \quad q^2 = \mathbf{v}\cdot\mathbf{v} \tag{1.3}$$

We find that the relations are definitely commutative with respect to interchange of the vectors $\mathbf{v}$ and $\boldsymbol{\omega}$. This yields a new invariant of the vortex lines and is expressed by the following theorem.

Theorem. Let μ be a twice continuously differentiable function constant along the vortex lines of the continuous barotropic gas flow

$$\boldsymbol{\omega}\cdot\nabla\mu = 0 \tag{1.4}$$

Then the quantity $\theta_\mu = \mathbf{v}\cdot\nabla\mu$ will also remain constant along the vortex lines

$$\boldsymbol{\omega}\cdot\nabla\,(\mathbf{v}\cdot\nabla\mu) = 0 \tag{1.5}$$

To prove the theorem we transform the left side of expression (1.5) using the well-known formula for the gradient of a scalar product. We obtain

$$\boldsymbol{\omega}\cdot\nabla\,(\mathbf{v}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot(\mathbf{v}\cdot\nabla)\nabla\mu + \boldsymbol{\omega}\cdot(\nabla\mu\cdot\nabla)\,\mathbf{v} \tag{1.6}$$

Applying the operator $\mathbf{v}\cdot\nabla$ to (1.4), we reduce the first term on its right-hand side to the form

$$\boldsymbol{\omega}\cdot\,(\mathbf{v}\cdot\nabla)\,\nabla\mu = -\nabla\mu\cdot\,(\mathbf{v}\cdot\nabla)\,\boldsymbol{\omega} \tag{1.7}$$

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Since $\omega = \text{rot}\, \mathbf{v}$, the second term can be written, after interchanging ω and $\nabla\mu$, as

$$\omega\cdot(\nabla\mu\cdot\nabla)\,\mathbf{v} = \nabla\mu\cdot(\omega\cdot\nabla)\,\mathbf{v}$$

Further, multiplying the Helmholtz-Friedman equation

$$(\mathbf{v}\cdot\nabla)\,\omega = (\omega\cdot\nabla)\,\mathbf{v} - \omega\,\text{div}\,\mathbf{v}$$

term by term by $\nabla\mu$ and taking (1.4) into account, we obtain

$$\omega\cdot(\nabla\mu\cdot\nabla)\,\mathbf{v} = \nabla\mu\cdot(\mathbf{v}\cdot\nabla)\,\omega \tag{1.8}$$

The equations (1.7), (1.8) show that the terms on the right-hand side of (1.6) differ only in sign and add upt to zero, which proves the theorem.

We note that the new invariant θ_μ has, unlike E_λ, the same form for flows of incompressible liquids and a compressible barotropic gas.

2. The theorem proved above and the properties of invariance (1.1), (1.3) together can be generalized to the case of adiabatic flow of a non-barotropic real gas. To do this we put in 1:1 corresponence the flow in question with parameters $(\mathbf{v}_*, p_*, \rho_*)$, and the isentropic (barotropic) flow $(\mathbf{v}, p, \rho)$ with the same configuration of the stream lines and the same pressure distribution, suing the transformation /4, 5/

$$\mathbf{v} = s_*^{-1/(2\varkappa)}\mathbf{v}_*,\ p = p_*,\ \rho = s_*^{1/\varkappa}\rho_* \tag{2.1}$$

where $s_* = p_*\rho_*^{-\varkappa}$ is the entropy function of the adiabatic flow constant along the stream lines and $\varkappa$ is the adiabatic index.

Relations (1.5), (1.1) and (1.3) hold in the field of isentropic flow, but we must put in (1.3)

$$H = i_0 = \frac{q^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho}$$

Introducing now the vector

$$\mathbf{B} = s_*^{-1/(2\varkappa)}\,\omega_* - \mathbf{v}_* \times \nabla\,(s_*^{-1/(2\varkappa)})\ \ (\omega_* = \text{rot}\,\mathbf{v}_*)$$

and the function μ_*, constant along vector lines of the vector $\mathbf{B}$, we find that the theorem proved in Sect.1 will be generalized as follows: in an adiabatic non-barotropic flow the quantity $\theta_\mu{}^* = s_*^{-1/(2\varkappa)}\,\mathbf{v}_*\cdot\nabla\mu_*$ remains constant together with μ_*, along vector lines of the vector $\mathbf{B}$.

According to (1.3), (2.1) the quantity $s_*^{-1/\varkappa}\,i_0{}^*$, where $i_0{}^*$ is the total enthalpy. will also be conserved along vector lines of the vector $\mathbf{B}$.

Note that these lines do not coincide with the vortex lines of the flow in question.

The Euler-Ertel theorem is generalized as follows: in an adiabatic, non-barotropic flow, the quantity $E_\lambda{}^* = (\mathbf{B}\cdot\nabla\lambda)/\rho_*$ remains constant together with λ along the stream lines.

3. The results of Sect.2 also hold for a gas with a more general equation of state, of the form $\rho_* = F\,(p_*)\,\Phi\,(\sigma_*)$.

The proof is analogous to that of Sect.2, but we use, in place of (2.1), the transformation /4/

$$\mathbf{v} = \Phi^{1/2}\,(\sigma_*)\,\mathbf{v}_*,\ \rho = \Phi^{-1}\,(\sigma_*)\,\rho_*$$

and $\mathbf{B} = \Phi^{1/2}\omega_* - \mathbf{v}_* \times \nabla\,(\Phi^{1/2})$.

4. Let us now obtain the equations describing the variation in the projection of the vorticity on the direction of the velocity in an ideal gas. We will write the vorticity vector in the form of the sum of two terms, one of which is oriented in a direction parallel, and the other perpendicular to the velocity vector

$$\omega = \omega_v + \omega_n \tag{4.1}$$

$$\omega_v = \tau\,(\omega\cdot\tau),\ \omega_n = \tau \times [\omega \times \tau],\ \tau = \mathbf{v}/q \tag{4.2}$$

$$\omega_v = \omega\cos\gamma,\ \ \omega_n = \omega\sin\gamma$$

where γ is the angle between the vectors ω and $\mathbf{v}$ in the plane Π passing through these vectors.

Crocco's theorem /6/

$$\omega \times \mathbf{v} = T\nabla\sigma - \nabla i_0 \tag{4.3}$$

connects the normal component of the vorticity with the change in entropy σ and the total enthalpy i_0 (T is the temperature).

We shall call the quantity $\Omega_v = \omega_v/(\rho q)$ the projection of the vorticity on the direction of the velocity of a compressible gas, and obtain an equation describing the change in the generalized projection of the vorticity on the direction of the velocity. By substituting (4.1) into the identity $\text{div}\,\omega = 0$ and using the equations of continuity, we find

$$\mathbf{v}\cdot\nabla\Omega_v = -\rho^{-1}\,\text{div}\,\omega_n \tag{4.4}$$

Applying the divergence operator to both sides of formula (4.2) for ω_n and using theorem (4.3) as well as the equation of motion of a gas in the Gromeka-Lamb form and the equation of state

$$\nabla \frac{q^2}{2} + \omega \times \mathbf{v} = -\frac{1}{\rho} \nabla p, \quad p = c_p \frac{\varkappa - 1}{\varkappa} \rho T$$

where c_p is the specific heat at constant pressure, we can show that in isentropic flow ($\sigma \equiv$ const)

$$\operatorname{div} \omega_n = \frac{2}{\rho q^2} (\omega_n \cdot \nabla p) \tag{4.5}$$

and in isoenergetic flow ($i_0 \equiv \text{const} = i_{00}$)

$$\operatorname{div} \omega_n = \frac{2 i_{00}}{c_p T \rho q^2} (\omega_n \cdot \nabla p) \tag{4.6}$$

Substituting (4.5), (4.6) into (4.4) and considering the relative variations in all the quantities, we finally obtain (M is the local Mach number).

$$\frac{d}{d\tau} \ln \Omega_v = -\frac{2 \operatorname{tg} \gamma}{\varkappa M^2} \frac{d}{dn} \ln p \quad (\sigma \equiv \text{const}) \tag{4.7}$$

$$\frac{d}{d\tau} \ln \Omega_v = -\frac{\varkappa + 1}{\varkappa} \varepsilon_0 \operatorname{tg} \gamma \frac{d}{dn} \ln p \quad (i_0 \equiv \text{const}) \tag{4.8}$$

$$\varepsilon_0 = \frac{\varkappa - 1}{\varkappa + 1} \left[1 + \frac{2}{(\varkappa - 1) M^2} \right]$$

$$d/d\tau \equiv \boldsymbol{\tau} \cdot \nabla, \; d/dn \equiv \mathbf{n} \cdot \nabla$$

Equations (4.7), (4.8) show that for Beltrani flow ($\omega \parallel \mathbf{v}$, $\gamma = 0, \pi$) the quantity Ω_v is constant along the stream lines, and this represents a generalization of the second Gromeka-Beltrani theorem /1/. The generalized projection of the vorticity on the direction of the velocity is conserved along the stream lines also in the case when the pressure gradient in a direction normal to the stream lines in the Π plane is equal to zero.

In the general case when the above conditions do not hold, the generalized projection of the vorticity on the direction of the velocity varies, in the exact formulation, along the stream lines in accordance with (4.7), (4.8). At the same time, using the explicit expressions for the right-hand sides of the equations obtained above we can show, that in certain approximate theories based on expansions in terms of small parameters, the right-hand sides of formulas (4.7), (4.8) are of a higher order of smallness than the left-hand sides. Consequently, the known integrals of the equations of gas dynamics are supplemented by a new integral expressing the conservation, along the stream lines, of the principal terms of the expansion of the generalized projection of the vorticity on the direction of the velocity.

Thus in the theory of a thin compressed layer in /7/ in the case of hypersonic flows past bodies, the limit transition $\varepsilon \to 0$ ($\varkappa \to 1$, $M_\infty \to \infty$), is used, where the parameter ε characterizes the ratio of the densities at the leading shock wave. At finite angles of attack we have $\varepsilon_0 \sim \varepsilon$. Therefore, according to (4.8) we find that in the problems of three-dimensional hypersonic flows past thin wings with low aspect ratio ($\operatorname{tg} \gamma \sim \varepsilon^{1/2}$, $d \ln p/dn \sim \varepsilon^{1/2}$) /8/ or past bodies of finite thickness ($\operatorname{tg} \gamma \sim 1$, $d \ln p/dn \sim 1$) /9/, the generalized projection of the vorticity on the direction of the velocity is constant along the stream lines. This made it possible to obtain an analytic solution to these three-dimensional non-linear problems. The important fact here is that this property of invariance is universal and holds for flows of real gas, as well as for flows of a gas reacting and radiating in the equilibrium and non-equilibrium mode /10, 11/. At the same time the property is not trivial, since the transverse component of the vorticity is not invariant.

In the case of a wing with moderate aspect ratio in hypersonic flow at large angles of attach ($\cos \alpha \sim \varepsilon^{1/2}$) we have $\varepsilon_0 \sim 1$. The projection of the vorticity on the direction of the velocity is still invariant as before, since $\operatorname{tg} \gamma \sim 1$, $d \ln p/dn \sim \varepsilon$ /12/.

Further, in the non-linear theory of small perturbations, for transonic flow past a thin wing with a large aspect ratio (the relative thickness and the aspect ratio are of the order of δ and $\delta^{-1/3}$ respectively, the number M of the incoming flow is nearly unity $|1 - M_\infty^2| \sim \delta^{2/3}$), using the known estimates /13, 14/ for the orders of magnitude of the quantities, we obtain that in the outer region of flow (the coordinate along the normal to the wing $y \sim \delta^{-1/3}$), as well as in the inner region near the wing, ($y \sim \delta$) $\varepsilon_0 \sim 1$, $\operatorname{tg} \gamma \sim \delta^{-1/3}$, $d \ln p/dn \sim \delta$. Therefore the projection of the vorticity on the direction of the velocity is preserved along the stream lines, but in the outer region the transverse component ω_n, is also preserved, while in the inner region the component of vorticity normal to the wing varies along the stream lines.

5. Equations (4,7), (4,8) can be generalized to the case of the non-steady motion of gas. For example, in place of (4.7) we can have (t is the time)

$$\frac{d}{dt}\ln\Omega_v = -\frac{2q\,\mathrm{tg}\,\gamma}{\varkappa M^2}\frac{\partial}{\partial n}\ln p - \frac{1}{\cos\gamma}\left(\mathbf{v}\cdot\frac{1}{q}\frac{\partial\mathbf{v}}{\partial t}\right)$$

$$\mathbf{v} = \boldsymbol{\omega}/\omega,\ d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$$

In the theory of a non-steady thin compressed layer the generalized projection of the vorticity on the direction of velocity is constant along the trajectories only in the case of flow past a thin wing of small aspect ratio, unlike the stationary case /9/.

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COMPATIBILITY EQUATIONS, STRESS FUNCTIONS, AND VARIATIONAL PRINCIPLES IN THE THEORY OF PRESTRESSED SHELLS*

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General statements of the theory of small deformations of thin shells with initial stresses are considered /1/. Compatibility equations are derived for the kinematic quantities, functions are found that satisfy the equilibrium equations identically, different variational principles of statics are formulated and proved, and distortion boundary conditions are obtained. The presence of initial stresses induces substantial singularities into these sections of the theory as compared with the linear theory of unstressed shells /2—5/. These singularities are due to the fact that the specific potential energy in the theory of small deformations of elastic shells with initial stresses depends not only on the tensors governing the change in metric and curvature of the surface, but also the rotation vector /1/.

The results obtained can be applied in shell stability problems as well as in the analysis of large shell deformations by the method of successive loadings when a linear problem of small deformations measured

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